Math 222 - Calculus 3

1 Series

1.1 Limits

Before examining convergence and divergence of series, a solid understanding of limits and the techniques to evaluate them is needed. In this section, we'll examine various techniques for evaluating limits and the underlying theorems that allow us to use them.

- **Composing limits**: In cases where we have a function that takes another function as an argument, such as $f(g(x))$, we have that

  $$\lim_{x\to\infty} f(g(x)) = f(\lim_{x\to\infty} g(x))$$

  as long as $f(x)$ is continuous at $\lim_{x\to\infty} g(x)$

- **Absolute values of limits that go to 0**: In the case of an alternating limit that goes to 0, we have:

  $$\lim_{x\to\infty} a_n = \lim_{x\to\infty} |a_n|$$

  if $\lim_{x\to\infty} |a_n| = 0$

- **Squeeze theorem**: If we have $b_n \leq a_n \leq B_n$ and $\lim_{n\to\infty} b_n = \lim_{n\to\infty} B_n = c$ then:

  $$\lim_{n\to\infty} a_n = c$$

- **L'Hospitals Rule**: To apply l'Hospitals, we need a functional equivalent of the sequence such that the nth term of the sequence is equal to $f(n)$. Then, if we have that $\lim_{x\to\infty} \frac{b_x}{a_x} = \frac{\infty}{\infty}$ and that a_x, b_x are continuously differentiable:

  $$\lim_{x\to\infty} \frac{b_x}{a_x} = lim_{x\to\infty} \frac{\frac{d}{dx} b_x}{\frac{d}{dx} a_x}$$

  - When we have $lim_{x\to\infty} a_x^{b_x} = 1^\infty$, you can use the property of logarithms to get $e^{lna^b} = e^{blna}$ and then bring the limit inside.

- **Rationalizing**: In cases where we have $a_x + b_x$ and no other methods seems to find a non divergent answer, it is sometimes useful to rationalize and multiply by $\frac{a_x - b_x}{a_x - b_x}$

1.2 Important Series

Before proceeding to the most important series tests, we need to examine 2 important types of series which will be useful later on for power series and comparison tests.

- **Geometric Series**: Series of the form $\sum_{n=0}^{\infty} r^n$. The series converges if $|r| \leq 1$. In particular:

  $$\sum_{n=0}^{\infty} r^n = \frac{1}{1-r}$$

- **P-Series**: Series of the form $\sum_{n=0}^{\infty} \frac{1}{n^P}$. The series converge for $P > 1$.

1.3 Series Tests

- **Test For Divergence**: If $\lim_{n\to\infty} a_n \neq 0$, then the series $\sum_{n=0}^{\infty} a_n$ diverges.
- **Integral Test**: If $f(n)$ is
  - Positive
  - Decreasing
  - Continuous

  Then $\int_0^{\infty} f(n)dx$ and $\sum_{n=0}^{\infty} a_n$ behave similarly.
- **Comparison Test**: For cases where you can compare a series to another one that you already know diverges/converges.
  - If $a_n \leq b_n$ after a certain point and $\sum_{n=0}^{\infty} b_n$ converges, then $\sum_{n=0}^{\infty} a_n$ converges.
  - If $a_n \geq b_n$ after a certain point and $\sum_{n=0}^{\infty} b_n$ diverges, then $\sum_{n=0}^{\infty} a_n$ diverges.
  - **Limit Comparison Test**: If $\lim_{n\to\infty} \frac{a_n}{b_n} = c \in (0, \infty]$ then either both series converge or they both diverge
- **Alternating Series Test**: If the series is of the form $\sum_{n=0}^{\infty}(-1)^n b_n$, then the series converges if:
  - $b_{n+1} \geq b_n$ starting at some point in the series
  - $\lim_{n\to\infty} b_n = 0$
- **Ratio Test**: Allows you to test for absolute convergence of certain series. Finding

$$\lim_{n\to\infty} |\frac{a_{n+1}}{a_n}| = L$$

  - $L < 1$ means the series is absolutely convergent
  - $L > 1$ means the series is divergent with $L = 1$ meaning the test is inconclusive
- **Root Test**: Allows for testing series of the form $\sum_{n=0}^{\infty} a_n^n$

$$\lim_{n\to\infty} \sqrt[n]{|a_n|} = L$$

  - $L < 1$ means the series is absolutely convergent
  - $L > 1$ means the series is divergent with $L = 1$ meaning the test is inconclusive

1.4 Power Series

Power series are series of the form $\sum_{n=0}^{\infty} c_n x^n$ or $\sum_{n=0}^{\infty} c_n(x-a)^n$ where each c_i is a constant and x is a number that can be plugged in.

- If the series is of the form $\sum_{n=0}^{\infty} c_n(x-a)^n$, we say that it is centered about a

The series will converge or diverge depending on the value of x being plugged in. There are 3 possibilities:

- The series converges only for $x = a$
- The series converges when $|x - a| < R$ where R is some number called the **radius of convergence**
  - If this is the case, the series can either converge or diverge at the endpoints with the whole interval where it converges being called the **interval of converge**
- The series converges for all $x \in \mathbb{R}$

To find the interval of convergence, start with the ratio test to find the radius then apply other tests to determine the convergence of the endpoints

1.5 Representation of Functions as Power Series

While the geometric series gives us the power series representation of $\frac{1}{1-x}$, we can represent other functions as power series. The following are a list of techniques for doing so

- **Algebraic Manipulations:** The idea here is to transform our original function into something of the form $g(x)\frac{1}{1-f(x)}$ where $g(x)$ is the numerator of the function. The equivalent power series will then be $g(x)\sum_{n=0}^{\infty} f(x)^n$.
  - The series converges for $|f(x)| < 1$ (check endpoints separately)
- **Partial Fraction decomposition:** If the denominator has a polynomial that is hard to transform into the required form, PFD can be performed to create a sum of functions which are easier to transform into the required form (the interval of convergence of the sum of the series will be the most restrictive interval of convergence amongst the series).
- **Powers of x-a:** Transform the function into the form $\frac{g(x)}{1-c(x-a)}$ to get the series $g(x)\sum_{n=0}^{\infty} c^n(x-a)^n$
- **Computing series:** If we have a series which is equal to the power series of a function that we know for a certain value of x that converges, we can compute the value of the series by computing the value of the function for that value of x.

Additionally, we can differentiate/integrate functions and their respective power series to find the relevant power series for their integral/derivative. To find the power series for the integral/derivative of a function, simply integrate/derive its power series. In doing so, two theorems are relevant:

- $\int \sum_{n=0}^{\infty} c^n(x-a) = \sum_{n=0}^{\infty} \int c^n(x-a)$ (Same for derivation).
- The radius of convergence of the integral/derivative of a power series is the radius of convergence of that power series (the interval, i.e. the endpoints, can change)

1.6 Maclaurin and Taylor Series

If we assume a function has a power series representation, we can successively find the coefficients for each term by taking a series of derivatives and plugging in the value of x that negates the term to the n-th power. Series of the form $x-a)^n$ found this way are called **Taylor series** and if $a = 0$ they are called **Maclaurin Series**. For a given function, the corresponding Taylor series is:

$$\sum_{n=0}^{\infty} \frac{f^n(a)}{n!}(x-a)^n$$

with its own radius/interval of convergence.

1.6.1 Important Maclaurin Series

- $e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$ with convergence for all $x \in \mathbb{R}$
- $sin(x) = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n+1)!} x^{2n+1}$ with convergence for all $x \in \mathbb{R}$
- $cos(x) = \frac{d}{dx} sin(x) = \frac{d}{dx} \frac{(-1)^n}{(2n+1)!} x^{2n+1} = \sum_{n=0}^{\infty} \frac{(-1)^n}{(2n)!} x^{2n}$ with convergence for all $x \in \mathbb{R}$
- $ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n}$

1.6.2 Finding Maclaurin/Taylor series

Usually, finding a Maclaurin/Taylor series for a function is done by either observing a pattern in successive derivations, by finding a way to modify the geometric series to represent it or by modifying known Taylor/Maclaurin series.

- **Ln:** We have that $ln(1+y) = \sum_{n=0}^{\infty} \frac{(-1)^n y^n}{n}$ for $|y| < 1$ and $y = 1$

– To find the Taylor series about a, we can use $lnx = ln(x-a+a) = ln(a(1+\frac{x-a}{a})) = lna + ln(1+\frac{x-a}{a})$and the above identity.

- **Terms up to certain power:** To find the power series up to a certain power of a function times/divided by another function, perform the multiplication/division and only gather the required terms

1.6.3 Uniqueness of Power Series

Power series representations of functions are **unique**. Thus, the power series of functions we computed by modifying the geometric series are in fact the Taylor series of those same functions. Thus, the Taylor series (when $a = 0$) of a polynomial is simply that polynomial.

- **Binomial Theorem:** $(x+y)^n = \sum_{n=0}^{k} = \binom{n}{k} x^{n-k} y^k$ where $\binom{n}{k} = \frac{n!}{k!(n-k!)}$

1.6.4 Applications of Taylor Series

- **Finding the n-th derivative:** To find the n-th derivative of a function without calculating it explicitly, transform the function into series form and find the coefficient of x^n. Then solve for the derivative in the formula for the Taylor/Maclaurin Series.
- **Integration**: Not all functions can be integrated using standard techniques. It is sometimes necessary to transform the function into a power series and integrate the series

2 Vector Functions

In this section, we look at vector functions instead of regular scalar functions, examining how we deal with them and their useful applications. A basic understanding of linear algebra is necessary, with a small recap of useful techniques/formulas being presented at the start.

2.1 Recap of Linear Algebra

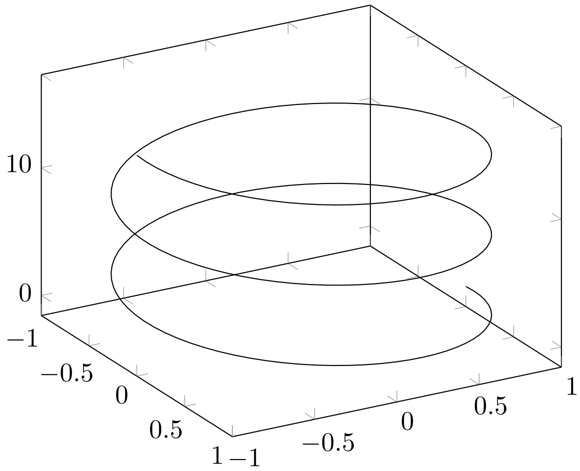
- **Cross Product:** $\vec{u} \times \vec{v} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \end{vmatrix}$
  - If $\vec{u}$ and $\vec{v}$ are non-zero vectors with $\vec{u} \times \vec{v} = \vec{0}$, then they are scalar multiples of each other
- **Equations of Lines:** To fully describe a line, we need the coordinates of a point and the direction of the line. $\bar{r}(t) = P + t\vec{v}$ where P is the coordinates of the point, $\vec{v}$ the direction vector and t a parameter. Thus, we can say $t \mapsto P + t\vec{v}$.
  - To find the equation of a line, given 2 points, simply subtract one point from the other to get the direction vector. We can also write the equation as $\bar{r}(t) = (1-t)P + tQ$
- **Equations of Planes:** To fully describe a plane, we need the coordinates of a point and a vector that is perpendicular to it. Thus, any point on the plane satisfies $(x-p) \bullet n = 0$. The formula can be written in scalar form as $n_x(x-p_x) + n_y(y-p_y) + n_z(z-p_z) = 0$
  - To find the equation of a plane given a normal or vice versa, either put the normal vector into the formula or deduce the normal vector using the coefficients of the equation.
  - To find the equation of a plane given a point and a line, check if the point is on the line. If it is not, then compute two non parallel vectors on the plane and find the cross product to get the normal.

2.2 Parametric curves

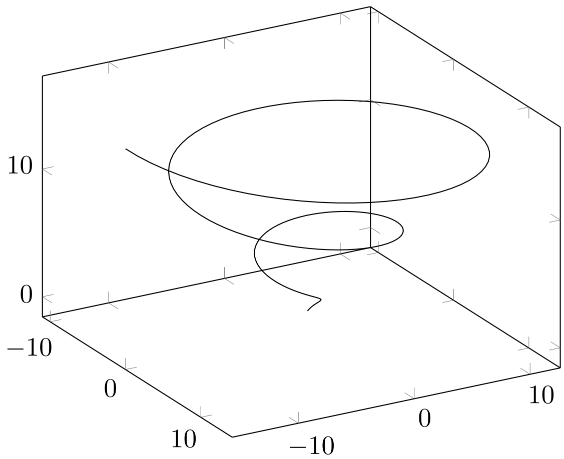
Parametric curves are curves described by vector functions instead of scalar functions. Thus, they vary relative to the value of a parameter that can affect any number of coordinates.

2.2.1 Important Parametric curves

- **Circle:** $r(t) =< rcost, rsint >$ where r is the radius
  - Works since $sin^2t + cos^2t = 1$ for all t.
  - The order of the function affects in which direction the circle is generated. By default it goes counterclockwise but switching the sign of $rsint$ can change that.
  - For the more general circle $(x - a)^2 + (y - b)^2 = r^2$, the corresponding parametrization is $r(t) =< a + rcost, b + rsint >$
- **Ellipse:** $r(t) =< acost, bsint >$ where a is the maximum value of the x coordinate.



- **Helix:** $r(t) =< cost, sint, t >$



- **Spiral:** $r(t) =< t, tcost, tsint >$

2.2.2 Parametrization of functions

We can parametrize any scalar function using $r(t) =< t, f(t) >$ but the reverse process is only possible when the curve passes the vertical line test.

- **Finding the scalar function of a curve:** If you have a vector function of the form $r(t) =< f(t), g(t) >$, solve $f(t)$ for t and then plug in what you find for t in $g(t)$. Make sure to consider the domain of the vector function when doing so.
- **Parametrizations of scalar functions are not unique:** A given scalar function can have multiple equivalent parametrizations (ex: $< f(t), [f(t)]^2 >$ and $< [f(t)]^2, [f(t)]^4 >$)

2.3 Calculus of Vector Functions

2.3.1 Limits

Taking the limit of a vector function is done component wise (find the limit of each component individually). If any of the limits is not defined, then the entire limit is not defined.

2.3.2 Derivatives

We define the derivative as $\frac{dr(t)}{dt} := lim_{h\to 0}\frac{r(t+h)-r(h)}{h}$. Using the aforementioned definition of limits of vector functions, we have that the derivative of a vector function is taken component wise.

$$\frac{dr(t)}{dt} =< f'(t), g'(t), h'(t) >$$

The derivative at t gives the slope of a line tangent to the curve at t.

- **Unit tangent vector:** We define $T(t) := \frac{r'(t)}{|r'(t)|}$

2 different parametrizations of the same scalar function will have different derivatives.

2.4 Integration

We define the definite integral as $\int_a^b r(t)dt := lim_{n\to\infty}\sum_{n=1}^{\infty} r(t_i^*)\Delta t$ where $\Delta t = \frac{b-a}{n}$. Due to the definition of the limit and the sum of vectors, we get that the integral is computed component wise.

$$\int_a^b r(t)dt =< \int_a^b f(t)dt, \int_a^b g(t)dt, \int_a^b h(t)dt >$$

FTC holds with this definition and thus if $R(t) = r'(t)$, $\int_a^b r(t)dt = R(b) - R(a)$. As for the indefinite integral, it is defined as:

$$\int r(t)dt = R(t) + c$$

2.5 Arc Length

For the case of a scalar function, we have that the length of the arc from $x = a$ to $x = b$ is $\int_a^b \sqrt{1 + [f'(x)]^2}dx$. The formula for the arc length of a parametric curve is derived in a similar fashion. We want to find the magnitude Δs of $r(t + h) - r(t)$ and then take the limit as h goes to 0. To do so, we take the square root of the sum of the squared derivatives of both components.

$$L = \int_a^b \sqrt{[f'(t)]^2 + [g'(t)]^2}dt = \int_a^b |r'(t)|dt$$

- **Arc Length Function:** The function that gives the arc length from a to a certain value of the parameter is $s(t) = \int_a^t |r'(s)|ds$

2.5.1 Reparametrizing WRT Arc Length

We can change the parameter of the vector function so that it is a function of arc length (meaning that it will give the position corresponding to having traveled a certain amount of units).

- Find t as a function of s (arc length)
- Replace t in $r(t)$ with $r(s) = r(t(s))$

An arc length parametrization has the property that $|r'(s)| = 1$. Intuitively this makes sense since it is saying that the magnitude of the rate of change of the vector function is 1 (if we increase the parameter by 1, we move 1 unit away). If $|r'(t)| = 1$, then t is an arc length parameter

2.6 Curvature

Curvature can be thought of geometrically as how far a curve is deviating from being a straight line at a given point. It can be computed using:

$$\kappa = |\frac{dT}{ds}| = |r''(s)|$$

where dT is the derivative of the unit tangent vector relative to the reparametrized vector function. Curvature can also be computed without reparametrizing the curve using either:

$$k = |\frac{T'(t)}{r'(t)}| = \frac{|r'(t) \times r''(t)|}{|r'(t)|^3}$$

- **Finding curvature of scalar equations:** Simply transform it into a vector function and calculate the curvature of the vector function
- The radius of a biggest circle tangent to a point in the curve is $\frac{1}{\kappa}$

2.7 Frenet Frames

A Frenet Frame is a system of orthogonal unit vectors (TNB).

- **T (Unit Tangent Vector):**$T(t) = \frac{r'(t)}{|r'(t)|}$
  - $T(t) \perp T'(t)$
- **N (Unit Normal Vector:** $N(t) = \frac{T'(t)}{|T'(t)|}$
  - $\frac{dT}{ds} = kN$ since N and $\frac{dT}{ds}$ are in the same direction (point towards the direction of concavity).
- **B (Binormal Vector):** $B(t) = T(t) \times N(t)$
  - If B is constant, the curve is purely planar (2-dimensional)

The largest circle tangent to a certain point of the curve is the **osculating circle**. It has radius $= \frac{1}{k}$. Additionally, for each P, the plane determined by T and N is the **osculating plane**.

2.8 Motion in Space

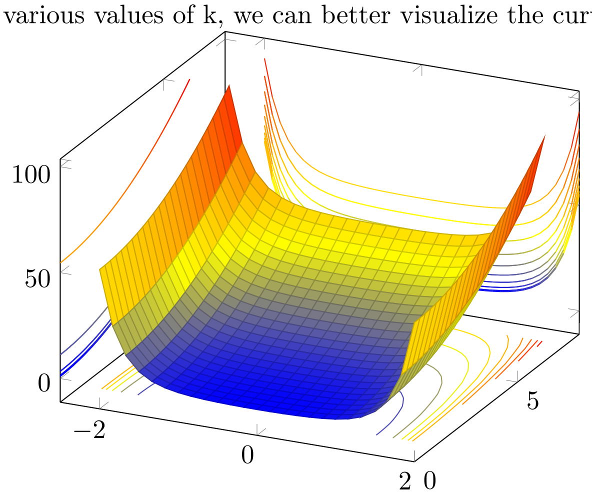
The aforementioned vector functions can be thought of as describing the motion of a particle in 3D space with the parameter t being time.

- $r(t)$ gives the position of the particle wrt to time
- $v(t) = r'(t)$ gives the velocity of the particle
  - $sp(t) = |v(t)|$ gives the speed of the particle
- $a(t) = v'(t) = r''(t)$ gives the acceleration of the particle
- **Tangential component of acceleration:** Can be calculated using $a_T = \frac{r' r''}{|r'|}$
- **Normal Component of acceleration:** $a_N = \frac{|r' \times r''|}{|r'|}$

3 Differentiation in Multivariable Calculus

3.1 Level Curves/Level Sets and 3D Shapes

It is considerably harder to visualize curves of 3D functions. The use of level sets/level curves are useful to help visualize these curves. A level set is defined as a a function of 2 variables with equation $f(x, y) = k$ where k belongs to the range of f. By looking at the corresponding level set for various values of k, we can better visualize the curve.



3.1.1 Important 3D Shapes

- **Elliptic Parabola:** $z = \frac{x^2}{a^2} + \frac{y^2}{b^2}$

3.2 Limits, Continuity and Differentiability

Unlike the 1D case where you can only approach a point from 2 directions, in the 2D case where x and y are approaching a certain coordinate there are infinitely many ways to approach the point. Thus, it is harder to prove the existence of a limit.

- **Proving it doesn't exist:** If you can prove that the limit when approaching from 2 different directions is different, then the limit does not exist. To do so, you can hold either x or y constant or put them as functions of the other variable and calculate the limit for those cases.
- **Proving it exists:** If you can split the function as $f(x)g(y)$ and the limits converge separately and the individual limits exist, then the whole limit is equal to the product of the individual limits

Continuity: $f(x, y)$ is continuous at (a, b) if $\lim_{(x,y)\to(a,b)} f(x, y)$ exists and is equal to $f(a, b)$

Differentiability:A stricter definition of differentiability is required since we need to ensure the function is differentiable from all directions. A function is differentiable at (a, b) if $lim_{(\Delta x,\Delta y)\to(0,0)} \frac{f(x+\Delta x,y+\Delta y)-f(x,y)}{\sqrt{(\Delta x)^2+(\Delta y)^2}}$ exists.

- A sufficient condition for differentiability is the existence of f_x and f_y at (a, b) and their being continuous for f to be differentiable at (a,b)

3.3 Partial Derivatives

The partial derivative with respect to a certain variable gives the rate of change of the function along that axis.

$$\frac{\partial f(a, b)}{\partial x} := lim_{h\to 0}\frac{f(a + h, b) - f(a, b)}{h}, h = \Delta x$$

with a similar definition for the partial derivative with respect to y.

$$\frac{\partial f}{x} = f_x = D_x f$$

To compute the derivative, treat y as a constant and differentiate normally

3.3.1 Higher order derivatives and mixed derivatives

- **Higher oder derivatives:** $\frac{\partial^2 f}{\partial x^2} \equiv f_{xx}$
- **Mixed derivatives:** $\frac{\partial^2 f}{\partial y \partial x} \equiv f_{xy}$ (Derive first with respect to x and then with respect to y)
- **Clairaut's theorem:** If f is such that $f_{xy} and f_{yx}$ are continuous at (a, b), then $f_{xy}(a, b) = f_{yx}(a, b)$.

3.4 Chain Rule:

If we have a function $f(x, y)$ where both x and y are functions of a single parameter, we can take the derivative of $f(x, y)$ relative to the parameter t using $\frac{df}{dt} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} = \nabla f \cdot r'(t)$.

The formula can be extended to functions of n variables of which each is a function of m parameters:

$$\frac{\partial f}{\partial t_i} = \sum \frac{\partial f}{\partial x_j}\frac{\partial x_j}{\partial t_i}$$

3.5 Implicit Differentiation

As in single variable calculus, it is possible to differentiate implicit functions by deriving both sides and isolating the required derivative. An easier method is to transform $f(x, y, z) = g(x, y, z)$ to $F = (x, y, z) = f(x, y, z) - g(x, y, z) = 0$. Then

$$\frac{\partial x}{\partial y} = \frac{-\frac{\partial F}{\partial y}}{\frac{\partial F}{\partial x}}$$

The same can be done for any combination of partial derivatives.

3.6 Directional Derivatives and Gradient Vectors

While we have been finding derivatives along the x and y axes, it is possible to find the derivative along other unit vectors.

Directional Derivative: The directional derivative of f at (a, b) in direction of $\bar{u}$ is

$$D_{\bar{u}}f(a, b) = \nabla f(a, b) \cdot \bar{u}$$

Gradient vector: $\nabla f(x, y) :=< f_x, f_y >$

- $D_{\bar{u}}f(a, b) = |\nabla f(a, b)| \cos\theta$
  - Thus, the directional derivative is maximized when $\bar{u}$ is in the same direction as $\nabla f(a, b)$. In other words, the gradient points in the direction of greatest rate of change of values of z at (a, b) (the rate is $|\nabla f(a, b)|$

3.7 Geometric Applications

- **Tangent Planes:** To find the equation of a plane tangent to a smooth surface defined by a function $z = f(\bar{x})$, we use the fact that the gradient acts as a normal vector (perpendicular to every tangent curve) giving us an equation of $(\bar{x} - \bar{p}) \cdot \nabla f(\bar{p}) = 0$
  - It is also possible to find 2 tangent vectors $(1, 0, f_x(\bar{p}))$ and $(0, 1, f_y(\bar{p}))$ and take their cross product to get the normal vector.
- **Max rate of change:** To find the max rate of change of $f(x, y)$ at a certain point, simply compute the gradient vector at that point and compute its magnitude (the actual vector will give you the direction where it occurs).

3.8 Maximization

3.8.1 Critical points

Identifying and classifying critical points is similar to single variable calculus. The first derivative helps us identify the critical points while the second derivative test allows us to classify them.

- **Identifying:** A critical point occurs when $\nabla f = 0$. Requires you to solve a system of equations.
- **Second derivative test:** To perform the test, we must first compute the Hessian matrix $\begin{bmatrix} f_{xx} & f_{xy} \\ f_{yx} & f_{yy} \end{bmatrix}$
  - **Minima:** $det > 0$ and $Tr > 0$
  - **Maxima:** $det > 0$ and $Tr < 0$
  - **Saddle:** $det < 0$
  - **Inconclusive:** $det = 0$. Means there are infinitely many critical points.

3.8.2 Maximization on Bounded Domain

Sometimes we want to find the maximum/minimum values of a function in a certain domain.

- **Triangle**:
  - Start by finding any minima/maxima in the interior by finding critical points.
  - Examine the values at the boundary of the triangle.
  - Find equations for the vertices of the triangle and rewrite the original function as a function of one variable and find minima/maxima of the new function using traditional single variable maximization.
  - Compare the values of all the critical points
- **Circle**:
  - Look at interior critical points
  - Parametrize the circle and replace the parameters in the initial function and maximize it using traditional methods. Also possible to rewrite x or y as a function of x or y and replace in the original function/maximize it.
  - Compare all values

3.8.3 Minimum/Maximum distance

To find the minimum/maximum distance, we need to examine the distance function. Since it'll often involve a root, we can instead maximize $f(x, y, z) = d^2(x, y, z)$. Since the points with the min/max square distance will have the min/max distance.

3.8.4 Constrained optimization with Lagrange multipliers

Since critical points occur when the gradient of the constrain is parallel to the gradient of the function that needs to be min/maxed, solving the following system of equations will allow us to find the critical points that when evaluated give the minimum/maximum.

$$\nabla f = \lambda \nabla g$$

$$g(x, y, z) = k$$

Some techniques for solving the system of equations include:

- Expressing each variable as a function of lambda and plugging into the constraint.
- Solving for lambda in each equation and equating the various equations.

4 Integration in Multivariable Calculus

4.1 Double Integrals on Rectangular Domains

Double integrals allow you to calculate the volume of a shape in a certain domain. The easiest way to define this domain is as a rectangle in which case we say the domain R is $R = [a, b] \times [c, d]$ or $(x, y) \in \mathbb{R}^2 : a \leq x \leq b, c \leq y \leq d$. To construct the double integral, we partition the rectangular domain into infinitely small squares of area $\Delta x \Delta y$ and multiply that area by the respective height, summing up the volumes obtained and taking the limit.

$$\int_R \int f(x, y) dA = lim_{n,m \to \infty} \sum_{i=1}^{n} \sum_{j=1}^{m} f(x*_{ij}, y*_{ij}) dx dy$$

To compute this, we simply find the inner integral and then integrate another time. **Fubini's Theorem:** If $f(x, y)$ is continuous on $R = [a, b] \times [c, d]$ then

$$\int_R \int f(x, y) dA = \int_a^b \int_c^d f(x, y) dy dx = \int_c^d \int_a^b f(x, y) dx dy$$

Making your life easier: If $f(x,y)$ is continuous on the domain and $f(x,y) = g(x)h(y)$ then you have that

$$\int_c^d \int_a^b f(x,y)dydx = \int_a^b g(x)dx \int_c^d h(y)dy$$

4.2 Double Integrals on General Domains

Unfortunately, life isn't always easy and it is sometimes required to integrate over weirdly shaped domains. To do so we enclose the domain with a rectangular one and modify the function so that it is equal to 0 when outside the domain. We then integrate over the rectangular domain using the new function.

$$\int_D \int_f (x,y)dA = \int_R \int F(x,y)dA$$

- **Type I Domain:** Found when the domain is delimited by a function above and below that's defined in terms of x and by $a \leq x \leq b$. Thus $D = \{(x,y), a \leq x \leq b, g_1(x) \leq y \leq g_2(x)\}$ and we have
$$\int_D \int_f (x,y)dA = \int_a^b \int_{g_1(x)}^{g_2(x)} f(x,y)dydx$$
- **Type II Domain:** Found when the domain is delimited by a function above and below that's defined in terms of y and by $a \leq y \leq b$. Thus $D = \{(x,y), a \leq y \leq b, g_1(y) \leq x \leq g_2(y)\}$ and we have
$$\int_D \int_f (x,y)dA = \int_a^b \int_{g_1(y)}^{g_2(y)} f(x,y)dxdy$$

To compute these, calculate the integral relative to the first variable and replace all instances of that variable by the functions found in the bounds like you would for a normal integral

4.3 Double Integrals Using Polar Coordinates

Instead of expressing coordinates as (x,y) pairs, we can express them as (r,θ) pairs where θ is the angle relative to the center in radians and r is the length of the vector. Doing so makes the integration over circular domains considerably easier. Thus, we have

$$\int_{D(x,y)} \int f(x,y)dA = \int_{D(r,\theta)} f(r\cos\theta, r\sin\theta)rdrd\theta$$

- To find the r boundary, replace x and y in the domain equation by $r\cos\theta$ and $r\sin\theta$ respectively and solve for the values of r that respect the equation.
- To find the θ boundary look at the graph and evaluate which angles are needed to trace the domain.